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Isocliny in spinor space and Wilson fermions

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Abstract

We show that Clifford algebras are closely related to the study of isoclinic subspaces of spinor spaces and, consequently, to the Hurwitz-Radon matrix problem. Isocliny angles are introduced to parametrize gamma matrices, i.e., matrix representations of the generators of finite-dimensional Clifford algebras $C(m,n)$. Restricting the consideration to the Clifford algebra $C(4,0)$, this parametrization is then applied to the study of Dirac traces occurring in Euclidean lattice quantum field theory within the hopping parameter expansion for Wilson fermions.

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1 Introduction

Clifford algebras [1]-[3] play an important role in a number of branches of mathematics and physics and are widely studied, therefore. However, the initial motivation for the present study has arisen from Euclidean lattice quantum field theory and, in particular, from the investigation of certain problems related to Wilson fermions which will be described in greater detail further below. Here, it suffices to mention that within the hopping parameter expansion for Wilson fermions Dirac traces (in $d = 2, 3, 4$ lattice dimensions related to the Clifford algebra $C(d, 0)$) have to be calculated whose detailed qualitative and quantitative understanding is of considerable theoretical importance. While in $d = 2$ a rather complete understanding exists, it is fairly incomplete for the physically more interesting dimensions $d = 3, 4$. Enquiries into the existent mathematical and physical literature showed that researchers in the field of Clifford algebras had not developed so far a framework which would have been sufficiently suited for the further exploration of this subject. In attempting to fill this gap soon it turned out that the relevant structures are not special to the Clifford algebras $C(d, 0)$, $d = 2, 3, 4$, but can be identified, at least in some preliminary way, for any finite-dimensional Clifford algebra $C(m, n)$. Therefore, the present study has two parts – a mathematical one which is focussed on Clifford algebras and another one which applies the obtained mathematical insight to a theoretical physics problem, i.e., the study of Wilson fermions.

In the first (mathematical) part (sect. 2), we consider an arbitrary finite-dimensional Clifford algebra $C(m, n)$ and show in subsect. 2.2, after having given the necessary notations in subsect. 2.1, how it is related to isoclinic subspaces of spinor space. This subject is considered here for the first time in the literature (some initial study of the subject by the present author for the Clifford algebra $C(3, 0)$ can be found in [4]). While the applied method is completely satisfactory for the Clifford algebra $C(4, 0)$ which we are primarily interested in from a theoretical physics point of view the investigation to be presented remains preliminary to some extent, as is explained in detail in subsect. 2.2, for the general case of an arbitrary finite-dimensional Clifford algebra $C(m, n)$. As one main result, we find that the eigenspaces of the generators of the Clifford algebra $C(m, n)$ to the eigenvalues $\lambda = \pm 1$, or $\lambda = \pm i$, are isoclinic to each other. Specifically, eigenspaces belonging to different generators are isoclinic to each other with an angle $\Theta = \pi/4$. This result is then further exploited in subsect. 2.3 to set up a parametrization of gamma matrices, i.e., matrix representations of the generators of the Clifford algebra $C(m, n)$, in terms of isocliny angles. According to Wong [5], the description of isoclinic subspaces is closely related to the Hurwitz-Radon matrix problem and, consequently, we exploit this link in subsect. 2.4 to connect Clifford algebra representations to the Hurwitz-Radon matrix problem. Finally, in subsect. 2.5 we discuss some formula which will turn out to be useful for the study of Wilson fermions.

The second (theoretical physics) part (sect. 3) of the article is devoted to the study of Wilson fermions in Euclidean lattice field theory. Fermions play a major role in most physically relevant models of quantum field theory, but their inclusion into numerical studies within lattice field theory remains to be hampered by the so-called sign problem, i.e., the appearance of contributions of alternating sign which have to be summed up. It is clear that in principle this sign problem is related to the anticommuting and spinorial character of fermionic variables. But, concentrating our attention onto Wilson fermions, not much is known in qualitative and quantitative respect concerning the emergence of sign factors in theories containing them. As mentioned above, the situation is understood rather satisfactory in 2 lattice dimensions [6], but an equivalent understanding in 3 and 4 dimensions is lacking. It is highly desirable for two reasons at least. In recent years, to a large extent based on the insight obtained into the sign problem for Wilson fermions in 2 dimensions exact equivalences between purely fermionic models of lattice quantum field theory and (multi-color) loop models (with a bending rigidity $\eta = 1/\sqrt{2} = \cos\Theta$) of standard statistical mechanics which can equivalently be understood as vertex models have been established [7]-[11] (also note [12, 13]). They represent a novel link between both branches of theoretical physics. These equivalent models are free of any sign problem and are interesting from an analytical as well as from a numerical point of view. It would clearly be of great theoretical interest to extend this novel link between lattice quantum field theory and standard statistical mechanics to models in more than 2 dimensions. Second, the recently proposed meron-cluster algorithm allows to effectively approach the numerical simulation of various models which exhibit a sign problem [14, 15]. However, it is based on a detailed understanding of the emergence of the sign factors in any model under consideration. This clearly adds further motivation for studying the sign problem for Wilson fermions. In the first subsection of sect. 3 we review the present state of knowledge concerning the calculation of Dirac traces for Wilson fermions. Based on the results obtained in sect. 2, in subsect. 3.2 we then show that in 3 lattice dimensions the calculation of Dirac traces can be performed in a very simple manner on the basis of different choices for the representation of gamma matrices. Subsect. 3.3 finally discusses the possible use of these representations for the calculation of Dirac traces in 4 dimensions. The article closes with some discussion in sect. 4.

2 Isoclinic subspaces of spinor spaces and Clifford algebras

2.1 Definitions and basics

The generators of a (finite-dimensional) Clifford algebra $C(m,n)$ obey the standard relation

$$\gamma_\mu \gamma_\nu \;+\; \gamma_\nu \gamma_\mu \;=\; 2\; g_{\mu\nu}\; \mathbf{1} \tag{2.1}$$

where $g_{\mu\nu}$, $\mu, \nu = 1, .., (m+n)$ are the elements of the diagonal matrix g with $g_{\mu\mu} = 1$ for $1 \leq \mu \leq m$ and $g_{\mu\mu} = -1$ for $m < \mu \leq n$ and $\mathbf{1}$ is the unit element of the Clifford algebra. They act as operators in a space V which is called the spinor space. For our purposes we consider the generators γ_μ as $2s \times 2s$, $s \in \mathbf{N}$, matrices (gamma matrices) with complex entries (for our purposes, it does not matter here if these matrices correspond to irreducible matrix representations of the Clifford algebra $C(m,n)$, or not). We choose the gamma matrices to be hermitian or antihermitian, respectively,

$$\gamma_\mu \;=\; g_{\mu\mu} \gamma_\mu^\dagger \; . \tag{2.2}$$

This is always possible (see, e.g., [16], chap. 1, §1, pp. 20-22, [17], chap. I, subsect. 4.3, iii), p. 11). We also define the following $2s \times 2s$ diagonal matrices ($\mathbf{1}_s$ denotes the $s \times s$ unit matrix).

$$\gamma_{E_p} \;=\; \begin{pmatrix} \mathbf{1}_s & 0 \\ 0 & -\mathbf{1}_s \end{pmatrix} \;=\; \gamma_E \quad , \quad \gamma_{E_p}^2 \;=\; \mathbf{1} \tag{2.3}$$

$$\gamma_{E_n} \;=\; i\gamma_E \quad , \quad \gamma_{E_n}^2 \;=\; -\mathbf{1} \tag{2.4}$$

Consider now the following eigenvalue equation for vectors ϕ in the (finite-dimensional) complex vector space $V = V_{\mathbf{C}}$ ($\simeq \mathbf{C}_{2s}$, $s \in \mathbf{N}$).

$$\gamma_\mu \phi \;=\; \lambda_\mu \phi \tag{2.5}$$

From eq. (2.1) immediately follows $\lambda_\mu = \lambda_{\mu\pm} = \pm 1$ for $1 \leq \mu \leq m$ and $\lambda_\mu = \lambda_{\mu\pm} = \pm i$ for $m < \mu \leq n$. The eigenspaces to the eigenvalues $\lambda_\mu = \pm 1, \pm i$ are of equal dimensionality s (multiplying eq. (2.5) by γ_ν, $\nu \neq \mu$ and taking into account eq. (2.1) one finds that for any eigenfunction ϕ to the eigenvalue λ, $\gamma_\nu \phi$ is an eigenfunction to the eigenvalue $-\lambda$). Any gamma matrix γ_μ can be diagonalized my means of an unitary transformation U_μ,

$$U_\mu \gamma_\mu U_\mu^\dagger \;=\; \gamma_\mu' \; , \tag{2.6}$$

and be brought either to the form of γ_{E_p}, (2.3) (for $1 \leq \mu \leq m$), or γ_{E_n}, (2.4) (for $m < \mu \leq n$).

We construct now orthogonal projectors $P_{\mu\pm}$,

$$P_{\mu\pm}^2 = P_{\mu\pm} , \tag{2.7}$$

$$P_{\mu\pm} = P_{\mu\pm}^\dagger , \tag{2.8}$$

onto the s-dimensional eigenspaces $V_{\mu\pm}$ of the generators γ_μ of the Clifford algebra $C(m,n)$. Explicitly, these projectors read

$$P_{\mu\pm} = \frac{1}{2}\left(\mathbf{1} + \frac{\gamma_\mu}{\lambda_{\mu\pm}}\right) . \tag{2.9}$$

The property (2.8) is fulfilled by virtue of eq. (2.2). In the same way, we also define orthogonal projection operators related to the eigenspaces $E_\pm$ of the matrices (2.3), (2.4).

$$P_{E_p\pm} = \frac{1}{2}\left(\mathbf{1} \pm \gamma_{E_p}\right) = P_{E\pm} \tag{2.10}$$

$$P_{E_n\pm} = \frac{1}{2}\left(\mathbf{1} \mp i\gamma_{E_n}\right) = P_{E\pm} \tag{2.11}$$

2.2 Isocliny

In the following, we will be interested in certain geometric, more precisely angular relations between the eigenspaces of the gamma matrices γ_μ, γ_E. For the purpose of the present investigation, we equip the spinor space $V = V_{\mathbf{C}}$ with the conventional (in general, *Spin* non-invariant) Hermitian form (product)

$$(a,b)_{\mathbf{C}} = \sum_{k=1}^{2s} \bar{a}_k b_k , \quad a,b \in V \tag{2.12}$$

($\bar{a}_k$ denotes the complex conjugate of a_k). Before we proceed further, in this context the following comment is due. In general, the inner product in a spinor space is appropriately chosen as being invariant under the *Spin* group related to the Clifford algebra under study ([18], app. B, §4, p. 307, [19]-[21], [22], chap. 2, sect. 2.6, p. 62, [23], [3], sect. 18, p. 231). However, the mathematical (geometrical) formalism we want to rely on in the present paper has only be developed so far in the literature for a scalar product which is related to the Hermitian form (2.12). Primarily, at the moment we are interested in using this formalism for deriving some information useful for certain theoretical physics problems. For those Clifford algebras we are interested in from a theoretical physics point of view the Hermitian product (2.12) is invariant under the corresponding *Spin* group (we are primarily interested in $C(4,0)$ which $Spin(4,0) = SU(2) \otimes SU(2)$ is related to). Therefore, without any further apology the following investigation is based on the Hermitian product (2.12). But, one has to keep in mind that all considerations below will have to be reconsidered on the basis of *Spin* invariant scalar products for the general case of a Clifford algebra $C(m,n)$, if one wants to treat the most general case on the appropriate footing in the future.

As already mentioned above, we are interested in the geometry of the set of s-dimensional subspaces given by the eigenspaces of γ_μ, γ_E in the $2s$-dimensional spinor space V. Their geometric relation can be characterized by means of the so-called *stationary angles (principal angles)* between any two of these subspaces which we will study in the present paper (for a comprehensive list of references on the subject of angles between subspaces see sect. II of [4]). In general, the relative position of two s-dimensional subspaces of an affine space which have a least one point in common can be characterized by means of s stationary angles θ_k, $0 \leq \theta_k \leq \frac{\pi}{2}$, $k = 1, \ldots, s$ (we consider the spinor space V as a vector space attached to the point 0 of some $2s$-dimensional affine space). There exist different approaches to calculate these stationary angles. Here, we rely on a different approach than that used in [4] in studying the Clifford algebra $C(3,0)$. The present approach turns out to be more convenient for the consideration of a general Clifford algebra $C(m,n)$. Consider the $2s \times 2s$ matrix

$$P_U P_W P_U \tag{2.13}$$

where P_U, P_W are orthogonal projectors onto the s-dimensional subspaces U, W of some $2s$-dimensional space V (of course, equally well one can consider the matrix $P_W P_U P_W$). The spectrum of the matrix (2.13) is given by $2s$ numbers s numbers of which are equal to 0 and the remaining s numbers are given by $\cos^2\theta_k$, $k = 1, \ldots, s$ ([24], chap. IV, §31, pp. 393/394, [25], chap. 1, problem 32, p. 68). If the s stationary angles θ_k, $k = 1, \ldots, s$, are equal to each other the two subspaces U and W are called *isoclinic* (to each other) and it holds ([26], sect. 2, p. 299, eq. (2.3), [27], p. 99, 2.3(i), [28], sect. 1, p. 481, (1.2)(5))

$$P_U P_W P_U \;\; = \;\; \cos^2\theta \; P_U \tag{2.14}$$

where θ is the isocliny angle ($\theta = \theta_1 = \ldots = \theta_s$). In other words, two subspaces $U, W \subset V$ (of equal dimension) are isoclinic to each other if the angle θ between any vector $x \in U$ and its orthogonal projection onto W (with respect to some inner product, here: the conventional Hermitian product) is independent of the vector x (for a comprehensive list of references on isoclinic subspaces see [4], sect. II) Note that the definition of angles in complex vector spaces requires special attention. For a corresponding review including the concept of isocliny angles see [29]. For our purposes it suffices to mention that the isocliny angle θ is independent of whether it is calculated in the complex vector space $V = V_{\mathbf{C}}$ itself or in the corresponding real vector space $V_{\mathbf{R}}$ equipped with an almost complex structure J and isometric to $V_{\mathbf{C}}$.

Having introduced some mathematical concepts in the preceding paragraph which will be required for the further discussion we can now proceed with our study of the Clifford algebra $C(m,n)$. Taking into account eq. (2.1), a simple calculation yields the following result in the case of an arbitrary Clifford algebra $C(m,n)$ ($\mu \neq \nu$, $k,l = \pm(1)$; incidentally, this result should be expected to apply also for infinite-dimensional Clifford algebras)[1].

$$P_{\mu k} P_{\nu l} P_{\mu k} \;=\; \tau \; P_{\mu k} \;, \quad \tau \;=\; \frac{1}{2} \tag{2.15}$$

By virtue of eq. (2.14) this means that any two eigenspaces of two different gamma matrices γ_μ, $\mu = 1, \ldots, (m+n)$, are isoclinic to each other with the isocliny angle

$$\Theta \;=\; \frac{\pi}{4} \;, \quad \cos^2 \Theta \;=\; \frac{1}{2} \;=\; \tau \tag{2.16}$$

(the two different eigenspaces of any generator γ_μ are of course isoclinic to each other with an angle $\theta = \frac{\pi}{2}$). This geometric relation has been considered in the literature for the first time in [4] in the case of the real Clifford algebra $C(3,0)$.

As we want to rely in the following on the results obtained by Wong [5], until further notice we consider not the complex vector space V itself (of (complex) dimension $2s$) but the real vector space $V_{\mathbf{R}}$ (of dimension $4s$) associated with V by means of an almost complex structure J and isometric to V. To be specific, we define the almost complex structure J by assigning any complex entry $z = a + ib$, $a, b \in \mathbf{R}$, of the gamma matrices γ_μ a 2×2 matrix as follows.

$$z \;=\; a + ib \;\;\longrightarrow\;\; \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \tag{2.17}$$

Accordingly, until further notice we consider the gamma matrices γ_μ as $4s \times 4s$ matrices with real entries. Now, from eq. (2.15) we may conclude that the set of eigenspaces of the gamma matrices γ_μ, $\mu = 1, \ldots, (m+n)$, forms a *set of mutually isoclinic* $2s$*-planes* in $\mathbf{R}_{4s}$ ([5], part I, sect. 3, p. 19). The elements of such a set are

[1] Note that eq. (2.15) is formally shape invariant with respect to transformations $\Lambda \in Spin(m,n)$: $\gamma_\mu \longrightarrow \gamma'_\mu = \Lambda \gamma_\mu \Lambda^{-1}$. However, as in general $\Lambda^{-1} \neq \Lambda^\dagger$ the projectors defined in eq. (2.9) are in the general case no longer orthogonal projectors because γ'_μ is no longer hermitian or antihermitian and, therefore, property (2.8) no longer applies.

pairwise isoclinic. Obviously, this set is a subset of some *maximal set of mutually isoclinic* $2s$*-planes* Φ. Such a maximal set is defined by the property that it is not a proper subset of a larger set of mutually isoclinic $2s$-planes. Furthermore, a (as large as possible) set ($\subset \Phi$) of subspaces of $\mathbf{R}_{4s}$ is called an *equiangular frame* if its elements are pairwise isoclinic to each other with the angle $\pi/4$ ([5], part I, sect. 5, p. 40). Consequently, the $(m+n)$ eigenspaces of the gamma matrices γ_μ to the eigenvalues $\lambda_{\mu+}$ or $\lambda_{\mu-}$, respectively, form subsets of certain equiangular frames. For the purpose of the present paper it appears to be useful to consider two disjoint equiangular frames Ω — one (Ω_+) related to the eigenspaces to the eigenvalue $\lambda_{\mu+}$, and the other one (Ω_-) related to the eigenspaces to the eigenvalue $\lambda_{\mu-}$. The following theorem by Wong will be helpful then (Φ is any maximal set of mutually isoclinic $2s$-planes in $\mathbf{R}_{4s}$; the indices have been changed to conform to the notation used in the present article): "If the angles between any $2s$-plane of Φ and the p $2s$-planes of an equiangular frame are θ_k $(1 \leq k \leq p)$, then

$$\sum_{k=1}^{p} \cos^2 2\theta_k \quad = \quad 1 \quad . \tag{2.18}$$

Conversely, given any set of p angles θ_k $(1 \leq k \leq p)$ such that $0 \leq \theta_k \leq \pi$ and $\sum \cos^2 2\theta_k = 1$, then there exists a unique $2s$-plane isoclinic to each of the p $2s$-planes of a given equiangular frame, making angles θ_k with them, and this $2s$-plane belongs to Φ" ([5], pt. I, sect. 5, p. 41, theorem 5.3 (b)). We denote this isoclinic $2s$-plane making angles θ_k $(1 \leq k \leq p)$ with the equiangular frame Ω_+ by A_+. It is always possible to give the related orthogonal projector P_{A_+} the form

$$P_{A_+} \quad = \quad P_{E_+} \tag{2.19}$$

([5], pt. I, sect. 3, p. 19). Together with A_+ also its orthogonal complement A_- ($\mathbf{R}_{4s} = A_+ \oplus A_-$) is an element of Φ. The $2s$-plane A_- makes the angles

$$\hat{\theta}_k \quad = \quad \frac{\pi}{2} - \theta_k \tag{2.20}$$

with the elements of Ω_+ ([5], pt. I, sect. 2, p. 16, lemma 2.2) and the orthogonal projector onto it reads in accordance with eq. (2.19)

$$P_{A_-} \quad = \quad P_{E_-} \quad . \tag{2.21}$$

From these considerations we immediately conclude that in view of eq. (2.14) the following equations apply.

$$P_{E_+} P_{\mu+} P_{E_+} \quad = \quad \cos^2 \theta_\mu \; P_{E_+} \tag{2.22}$$

$$P_{E_-} P_{\mu+} P_{E_-} \quad = \quad \cos^2 \hat{\theta}_\mu \; P_{E_-} \; = \; \sin^2 \theta_\mu \; P_{E_-} \tag{2.23}$$

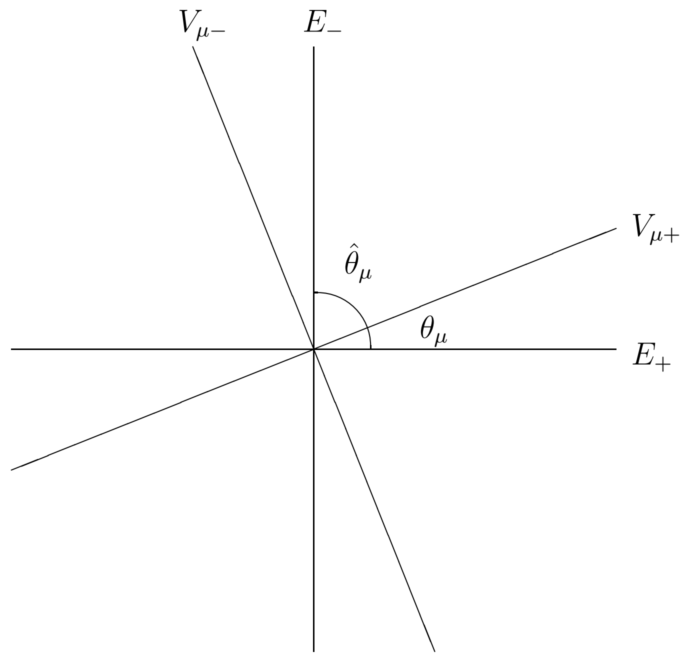

Figure 1: Geometry of the eigenspaces $V_{\mu\pm}$ of the gamma matrix γ_μ and of the subspaces $E_\pm$

From eqs. (2.22), (2.23), by simple algebra the analogous equations which are expected to apply for symmetry reasons follow.

$$P_{E_-} P_{\mu -} P_{E_-} \;=\; \cos^2\theta_\mu \; P_{E_-} \tag{2.24}$$

$$P_{E_+} P_{\mu -} P_{E_+} \;=\; \cos^2\hat{\theta}_\mu \; P_{E_+} \;=\; \sin^2\theta_\mu \; P_{E_+} \tag{2.25}$$

In other words, the isoclinic $2s$-plane A_- makes angles $\hat{\theta}_k$ $(1 \le k \le p)$ with the elements of the equiangular frame Ω_-. For a two-dimensional illustration of the eqs. (2.22)-(2.25) see fig. 1.

2.3 Parametrization of gamma matrices in terms of isocliny angles

Having established the equations (2.22)-(2.25) from now on we can return in our consideration to the original spinor space V and derive from these equations further conclusions. To begin with, by taking the sum of the left and right hand sides of the eqs. (2.22), (2.24) one can obtain the following relation by inserting eqs. (2.9)-(2.11) (the same result follows from eqs. (2.23), (2.25); in the following equations me omit the unit matrices $\mathbf{1}$, $\mathbf{1}_s$ whenever it seems to be appropriate).

$$\gamma_\mu \gamma_E \ + \ \gamma_E \gamma_\mu \ = \ 2\lambda_{\mu+} \cos 2\theta_\mu \tag{2.26}$$

Taking the difference of the left and right hand sides of the eqs. (2.22), (2.24) yields

$$\begin{aligned} W_{\mu+} \gamma_\mu W_{\mu+}^\dagger \ &= \ \lambda_{\mu+} \cos^2 \theta_\mu \ \gamma_E \ , & (2.27) \\ W_{\mu+} \ &= \ \frac{1}{2} \left(\mathbf{1} + \frac{\gamma_E \gamma_\mu}{\lambda_{\mu+}} \right) \ . & (2.28) \end{aligned}$$

Correspondingly, taking the difference of the left and right hand sides of the eqs. (2.23), (2.25) yields

$$\begin{aligned} W_{\mu-} \gamma_\mu W_{\mu-}^\dagger \ &= \ \lambda_{\mu-} \sin^2 \theta_\mu \ \gamma_E \ = \ \lambda_{\mu-} \cos^2 \hat{\theta}_\mu \ \gamma_E \ , & (2.29) \\ W_{\mu-} \ &= \ \frac{1}{2} \left(\mathbf{1} + \frac{\gamma_E \gamma_\mu}{\lambda_{\mu-}} \right) \ . & (2.30) \end{aligned}$$

As one sees, the equations related to $W_{\mu+}$ and $W_{\mu-}$ just differ in the factor $\lambda_{\mu\pm}$ and the use of the angles θ_μ and $\hat{\theta}_\mu$. From the eqs. (2.28), (2.30) one finds

$$W_{\mu-} \ + \ W_{\mu+} \ = \ \mathbf{1} \ . \tag{2.31}$$

The eqs. (2.28), (2.30) entail

$$\begin{aligned} W_{\mu\pm} \gamma_E \ &= \ \gamma_E W_{\mu\pm}^\dagger \ , & (2.32) \\ \gamma_\mu W_{\mu\pm} \ &= \ W_{\mu\pm}^\dagger \gamma_\mu \ , & (2.33) \end{aligned}$$

and

$$\gamma_\mu \ = \ \lambda_{\mu\pm} \ \gamma_E \ (2W_{\mu\pm} - \mathbf{1}) \ . \tag{2.34}$$

We can now infer further properties of the matrices $W_{\mu\pm}$ from those of the gamma matrices γ_μ. Taking into account eqs. (2.26), (2.28), (2.30) we find ($\theta_{\mu+} = \theta_\mu$, $\theta_{\mu-} = \hat{\theta}_\mu = \frac{\pi}{2} - \theta_\mu$)

$$W_{\mu\pm} W_{\mu\pm}^\dagger \ = \ W_{\mu\pm}^\dagger W_{\mu\pm} \ = \ \left\{ \begin{array}{c} \cos^2 \theta_\mu \\ \sin^2 \theta_\mu \end{array} \right\} \ = \ \cos^2 \theta_{\mu\pm} \ . \tag{2.35}$$

From $\gamma_\mu^2 = g_{\mu\mu}$ (eq. (2.1)) and eqs. (2.34), (2.35) we obtain

$$W_{\mu\pm} \ + \ W_{\mu\pm}^\dagger \ = \ 2 \left\{ \begin{array}{c} \cos^2 \theta_\mu \\ \sin^2 \theta_\mu \end{array} \right\} \ = \ 2 \ \cos^2 \theta_{\mu\pm} \ . \tag{2.36}$$

Also, from eq. (2.1) we find after inserting eq. (2.34) and using eq. (2.36)

$$\begin{aligned} W^\dagger_{\mu\pm} W_{\nu\pm} \ + \ W^\dagger_{\nu\pm} W_{\mu\pm} \ &= \ W_{\mu\pm} W^\dagger_{\nu\pm} \ + \ W_{\nu\pm} W^\dagger_{\mu\pm} \\ &= \ \cos^2\theta_{\mu\pm} \ + \ \cos^2\theta_{\nu\pm} \ + \ \frac{\delta_{\mu\nu} - 1}{2} \ . \end{aligned} \tag{2.37}$$

For the further discussion, it turns out to be convenient to represent the matrices $W_{\mu\pm}$ as block matrices. We write

$$W_{\mu\pm} \ = \ \begin{pmatrix} w_{\mu\pm 11} & w_{\mu\pm 12} \\ w_{\mu\pm 21} & w_{\mu\pm 22} \end{pmatrix} \tag{2.38}$$

where the submatrices $w_{\mu\pm kl}$ are $s \times s$ matrices. From eq. (2.32), we immediately find

$$w^\dagger_{\mu\pm 21} \ = \ - \, w_{\mu\pm 12} \ , \tag{2.39}$$

$$w^\dagger_{\mu\pm 11} \ = \ w_{\mu\pm 11} \quad , \quad w^\dagger_{\mu\pm 22} \ = \ w_{\mu\pm 22} \ . \tag{2.40}$$

Eq. (2.36) then entails

$$w_{\mu\pm 11} \ = \ w_{\mu\pm 22} \ = \ \left\{ \begin{array}{c} \cos^2\theta_\mu \\ \sin^2\theta_\mu \end{array} \right\} \ = \ \cos^2\theta_{\mu\pm} \ . \tag{2.41}$$

Consequently, the $W_{\mu\pm}$ matrices assume the explicit form (for convenience, we introduce the notation $w_{\mu\pm 12} = w_{\mu\pm}/2$).

$$W_{\mu\pm} \ = \ \frac{1}{2} \begin{pmatrix} 2\cos^2\theta_{\mu\pm} & w_{\mu\pm} \\ -w^\dagger_{\mu\pm} & 2\cos^2\theta_{\mu\pm} \end{pmatrix} \ . \tag{2.42}$$

By virtue of eq. (2.31) holds

$$w_{\mu+} \ = \ - \, w_{\mu-} \tag{2.43}$$

and eq. (2.35) yields

$$w_{\mu\pm} w^\dagger_{\mu\pm} \ = \ w^\dagger_{\mu\pm} w_{\mu\pm} \ = \ \sin^2 2\theta_\mu \ = \ \sin^2 2\hat{\theta}_\mu \ . \tag{2.44}$$

The gamma matrices γ_μ read in accordance with eq. (2.34)

$$\gamma_\mu \ = \ \lambda_{\mu\pm} \begin{pmatrix} \cos 2\theta_{\mu\pm} & w_{\mu\pm} \\ w^\dagger_{\mu\pm} & -\cos 2\theta_{\mu\pm} \end{pmatrix} \ . \tag{2.45}$$

This is the parametrization of gamma matrices in terms of isocliny angles mentioned in the Introduction. By virtue of eqs. (2.20), (2.43), this representation is independent of whether on the r.h.s. of eq. (2.45) the upper or lower signs are chosen.

For future purposes, we express the following product of $W_{\mu\pm}$ matrices in terms of the matrices $w_{\mu\pm}$ $(k,l=\pm(1))$.

$$T_{(\mu k,\nu l)} \;=\; W_{\mu k}W^{\dagger}_{\nu l} \;=\; \begin{pmatrix} t_{(\mu k,\nu l)11} & t_{(\mu k,\nu l)12} \\ t_{(\mu k,\nu l)21} & t_{(\mu k,\nu l)22} \end{pmatrix} \tag{2.46}$$

$$t_{(\mu k,\nu l)11} \;=\; \cos^2\theta_{\mu k}\cos^2\theta_{\nu l} \;+\; \frac{1}{4}\; w_{\mu k}w^{\dagger}_{\nu l} \tag{2.47}$$

$$t_{(\mu k,\nu l)22} \;=\; \cos^2\theta_{\mu k}\cos^2\theta_{\nu l} \;+\; \frac{1}{4}\; w^{\dagger}_{\mu k}w_{\nu l} \tag{2.48}$$

$$t_{(\mu k,\nu l)12} \;=\; -\,t^{\dagger}_{(\mu k,\nu l)21} \;=\; \frac{1}{2}\;\left(\cos^2\theta_{\nu l}\; w_{\mu k} \;-\; \cos^2\theta_{\mu k}\; w_{\nu l}\right) \tag{2.49}$$

Inserting eqs. (2.46)-(2.49) into (2.37) and taking into account eq. (2.43) we finally obtain

$$\begin{aligned} w^{\dagger}_{\mu k}w_{\nu l} \;+\; w^{\dagger}_{\nu l}w_{\mu k} \;&=\; w_{\mu k}w^{\dagger}_{\nu l} \;+\; w_{\nu l}w^{\dagger}_{\mu k} \\ &=\; 2\left[k\; l\; \delta_{\mu\nu} - \cos 2\theta_{\mu k}\cos 2\theta_{\nu l}\right]\;. \end{aligned} \tag{2.50}$$

Introducing the matrix $\widetilde{w}_{\mu\pm}$ by writing

$$w_{\mu\pm} \;=\; 2\;\cos^2\theta_{\mu\pm}\;\widetilde{w}_{\mu\pm} \tag{2.51}$$

one finds that the eqs. (2.44) and (2.50) expressed in terms of $\widetilde{w}_{\mu\pm}$ exactly agree with the eqs. (3.1) and (3.2) in [5], part I, sect. 3, p. 20, lemma 3.1 (in the notation of Wong the matrices $\widetilde{w}_{\mu\pm}$ describe isoclinic subspaces with an isocliny angle $\theta_{\mu\pm}$ relative to the subspace E_+).

2.4 Clifford algebras and the Hurwitz-Radon matrix problem

From eqs. (2.27), (2.29) we recognize that any gamma matrix γ_μ can be diagonalized in accordance with eq. (2.6) my means of the unitary matrix (we disregard here the case $\theta_{\mu\pm}=\frac{\pi}{2}$ which needs to be discussed separately)

$$U_{\mu\pm} \;=\; \frac{W_{\mu\pm}}{\cos\theta_{\mu\pm}}\;. \tag{2.52}$$

Using eqs. (2.28), (2.30) and (2.2) one finds the following representations of the gamma matrices in terms of the matrices $U_{\mu\pm}$.

$$\gamma_\mu \;=\; \lambda_{\mu\pm}\left(U^{\dagger}_{\mu\pm}\right)^2\gamma_E \;=\; \lambda_{\mu\pm}U^{\dagger}_{\mu\pm}\gamma_E U_{\mu\pm} \;=\; \lambda_{\mu\pm}\gamma_E U^2_{\mu\pm} \tag{2.53}$$

Now, choose $\mu = \mu_0$ and elevate U_{μ_0+} to a unitary matrix by means of which all gamma matrices related to the Clifford algebra $C(m,n)$ are being transformed. Then, after some calculation taking into account eqs. (2.28), (2.30) and (2.26) one finds (we apply here the inverse transformation compared with eq. (2.6))

$$\begin{aligned} \gamma_\nu &= U^\dagger_{\mu_0+} \gamma'_\nu U_{\mu_0+} \\ &= \gamma'_\nu \ + \lambda_{\nu+} \frac{\cos 2\theta_\nu}{\cos\theta_{\mu_0}} \ \gamma_E U_{\mu_0+} \ , \quad \nu \neq \mu_0 \ . \end{aligned} \tag{2.54}$$

As $\gamma'_{\mu_0} = \lambda_{\mu_0+}\gamma_E$ ($\theta'_{\mu_0} = 0$), from eq. (2.18) we immediately conclude that

$$\theta'_\nu \ = \ \Theta \ = \ \frac{\pi}{4} \ , \quad \nu \neq \mu_0 \ . \tag{2.55}$$

Eq. (2.54) then yields the following relation for the submatrices $w_{\nu+}$.

$$w_{\nu+} \ = \ w'_{\nu+} \ + \ \frac{\cos 2\theta_\nu}{2\cos^2\theta_{\mu_0}} \ w_{\mu_0+} \ , \quad \nu \neq \mu_0 \tag{2.56}$$

Inserting this relation into eq. (2.44) we find

$$\begin{aligned} w'^\dagger_{\nu+} \ w_{\mu_0+} \ + \ w^\dagger_{\mu_0+} \ w'_{\mu+} \ &= \ w'_{\nu+} \ w^\dagger_{\mu_0+} \ + \ w_{\mu_0+} \ w'^\dagger_{\nu+} \\ &= \ - \ 2\cos 2\theta_\nu \ , \quad \nu \neq \mu_0 \ . \end{aligned} \tag{2.57}$$

Consequently, once a set of gamma matrices γ'_μ, $\mu = 1, \ldots, (m+n)$, with $\theta'_{\mu_0} = 0$ and $\theta'_\nu = \frac{\pi}{4}$ for $\nu \neq \mu_0$ is given the choice of the matrix w_{μ_0+} uniquely determines the set of gamma matrices γ_μ which exhibit the isocliny angles θ_μ (and which are to be determined from the eq. (2.57) and for $\mu = \mu_0$ from eq. (2.44)). According to eq. (2.50), the $(m+n-1)$ matrices $w'_{\mu+}$, $\mu \neq \mu_0$ (of course, $w'_{\mu_0+} = 0$), obey the equation

$$\begin{aligned} w'^\dagger_{\mu+} \ w'_{\nu+} \ + \ w'^\dagger_{\nu+} \ w'_{\mu+} \ &= \ w'_{\mu+} \ w'^\dagger_{\nu+} \ + \ w'_{\nu+} \ w'^\dagger_{\mu+} \\ &= \ 2\delta_{\mu\nu} \ , \quad \mu, \nu \neq \mu_0 \ . \end{aligned} \tag{2.58}$$

Eq. (2.58) together with $w'_{\mu+} w'^\dagger_{\mu+} = \mathbf{1}_s$ (cf. eq. (2.44)) represent the *unitary Hurwitz-Radon matrix problem* for the matrices $w'_{\mu+}$ [30], subsection 1.4, p. 25, also see [5], part II, p. 67 and [31]. For a different approach relating (real) Clifford algebras to (generalized) Hurwitz-Radon matrix problems see [32]-[41].

2.5 Some useful formula

In this final subsection we focus our attention onto the matrix product (2.46). Let us start with the observation that the matrices $t_{(\mu k,\nu l)ij}$, $i, j = 1, 2$, obey the equation

$$t_{(\mu k,\nu l)ij} \ t^\dagger_{(\mu k,\nu l)ij} \ = \ t^\dagger_{(\mu k,\nu l)ij} \ t_{(\mu k,\nu l)ij}$$

$$
\begin{aligned}
&= \frac{1}{2}\left[1 + k\ l\ (-1)^{i+j}\delta_{\mu\nu}\right]\ \cos^2\theta_{\mu k}\ \cos^2\theta_{\nu l} \\
&= \tau\left[1 + k\ l\ (-1)^{i+j}\delta_{\mu\nu}\right]\ \cos^2\theta_{\mu k}\ \cos^2\theta_{\nu l}\ . \qquad (2.59)
\end{aligned}
$$

This can easily be checked by taking into account eq. (2.50). The last line can be seen to apply by writing

$$
P_{\mu\pm} = U^\dagger_{\mu\pm} P_{E_+} U_{\mu\pm} \qquad (2.60)
$$

and inserting it into eq. (2.15). Now, for later purposes let us further study the matrix $t_{(\mu k,\nu l)11}$ for $\mu \neq \nu$. According to eq. (2.59) the matrix

$$
\begin{aligned}
\check{t}_{(\mu k,\nu l)} &= \frac{t_{(\mu k,\nu l)11}}{\sqrt{\tau}\ \cos\theta_{\mu k}\ \cos\theta_{\nu l}} \\
&= \sqrt{2}\ \cos\theta_{\mu k}\cos\theta_{\nu l} + \frac{w_{\mu k}w^\dagger_{\nu l}}{2\sqrt{2}\cos\theta_{\mu k}\ \cos\theta_{\nu l}} \qquad (2.61)
\end{aligned}
$$

is unitary. Using eq. (2.50), one can easily check that this matrix can be represented as follows ($\alpha_{(\mu k,\nu l)} \in \mathbf{R}$). For convenience, we introduce here an antisymmetric sign factor $f_{\mu\nu} = -f_{\nu\mu} = \pm 1$ whose sign can later be arranged arbitrarily to simplify explicit expressions.

$$
\begin{aligned}
\check{t}_{(\mu k,\nu l)} &= \mathrm{e}^{\left[\alpha_{(\mu k,\nu l)}\ I_{(\mu k,\nu l)}\right]} \\
&= \cos\alpha_{(\mu k,\nu l)} + I_{(\mu k,\nu l)}\ \sin\alpha_{(\mu k,\nu l)}\ , \quad \mu \neq \nu \qquad (2.62) \\
I_{(\mu k,\nu l)} &= k\ l\ f_{\mu\nu}\ \sqrt{2}\ \frac{\left[w_{\mu k}w^\dagger_{\nu l} + \cos 2\theta_{\mu k}\cos 2\theta_{\nu l}\right]}{\sqrt{-\cos 4\theta_{\mu k} - \cos 4\theta_{\nu l}}}\ , \\
I^2_{(\mu k,\nu l)} &= -\ \mathbf{1}_s \qquad (2.63) \\
\sin\alpha_{(\mu k,\nu l)} &= k\ l\ f_{\mu\nu}\ \frac{\sqrt{-\cos 4\theta_{\mu k} - \cos 4\theta_{\nu l}}}{4\ \cos\theta_{\mu k}\ \cos\theta_{\nu l}} \qquad (2.64) \\
\cos\alpha_{(\mu k,\nu l)} &= \frac{1 + \cos 2\theta_{\mu k} + \cos 2\theta_{\nu l}}{2\sqrt{2}\ \cos\theta_{\mu k}\ \cos\theta_{\nu l}} \qquad (2.65)
\end{aligned}
$$

It seems to be worth emphasizing that the angle $\alpha_{(\mu k,\nu l)}$ only depends on the isocliny angles $\theta_{\mu k}$ and not on any details of the choice of the matrices $w_{\mu k}$. For $w_{\mu k} = w'_{\mu k}$, i.e., $\theta_{\mu k} = \Theta = \frac{\pi}{4}$, $\mu \neq \mu_0$, from the above equations one immediately finds

$$
\check{t}_{(\mu k,\nu l)} = \mathrm{e}^{\left[\frac{\pi}{4}\ w'_{\mu k}w'^\dagger_{\nu l}\right]}\ , \quad \mu \neq \nu,\ \mu,\nu \neq \mu_0\ . \qquad (2.66)
$$

Finally, note that the case $\theta_\mu = \frac{\pi}{2}$ requires some special, separate consideration.

3 Dirac traces for Wilson fermions

3.1 The problem

For a number of reasons, Wilson fermions are frequently applied in lattice field theory calculations (see, e.g., [42, 43]). The sign problem for fermions already shows up for free fermions and for simplicity here we restrict our consideration to these. The partition function Z_Λ for free Wilson fermions on a d-dimensional (hyper-) cubic lattice Λ is given by

$$Z_\Lambda \;=\; \int D\psi D\bar{\psi} \;\; \mathrm{e}^{-S} \quad , \tag{3.1}$$

where $D\psi D\bar{\psi}$ denotes the multiple Grassmann integration on the lattice. The action S is defined by

$$\begin{aligned} S \;=\; \sum_{x\in\Lambda} \Bigg(& \sum_{\mu} \left(\bar{\psi}(x+e_\mu) P_{\mu+} \psi(x) + \bar{\psi}(x) P_{\mu-} \psi(x+e_\mu) \right) \\ & - M \bar{\psi}(x)\psi(x) \Bigg) \end{aligned} \tag{3.2}$$

In eq. (3.2) the Wilson parameter r has been set to 1. We are considering here Euclidean lattice field theory, therefore, the Clifford algebra relevant for the study of Wilson fermions is $C(d,0)$ ($d = 2,3,4$). Within the hopping parameter expansion the partition function Z_Λ can be written as follows [43], chap. 12, p. 165.

$$Z_\Lambda \;=\; M^{|\Lambda|} \; \exp\left[-\sum_{L} \, (2\kappa)^{|L|} \; \mathrm{tr}\, \Gamma_L \right] \tag{3.3}$$

Here, $\kappa = 1/2M$ is the hopping parameter, L is any single closed loop configuration on the lattice Λ, $|L|$ denotes its length and

$$\Gamma_L \;=\; \prod_{l=1}^{|L|} P_{\mu_l k_l} \;=\; P_{\mu_1 k_1} P_{\mu_2 k_2} \cdots P_{\mu_{|L|-1} k_{|L|-1}} P_{\mu_{|L|} k_{|L|}} \tag{3.4}$$

is the path-ordered product of the projection matrices displayed in eq. (3.2) (in the hopping parameter expansion to each projector $P_{\mu\pm}$ corresponds a (directed) line between two neighboring points in the d-dimensional (hyper-) cubic lattice in the direction μ). In the following we will just be interested in the calculation of the trace of an arbitrary such matrix Γ_L. This problem has systematically been studied in [6].

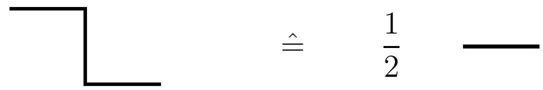

Figure 2: "Kink rule" to be used in calculating traces of products of projection operators $P_{\mu\pm}$ which occur in the hopping parameter expansion of the partition function Z_Λ, eq. (3.3) (in this figure arrows have been omitted for simplicity). η is the weight assigned to a corner in the loop model picture.

For 2 lattice dimensions, the problem has been solved in a completely satisfactory way where one finds using $2s \times 2s$ gamma matrices, $s = 1, 2$, and applying free boundary conditions that ([6], p. 1131, eq. (15); also see [9], p. 489, eq. (50); B_L is the analogue of the Kac-Ward sign factor met in the study of the two-dimensional Ising model [44], [45], chap. 5, sect. 5.4, p. 136)

$$\mathrm{tr}\,\Gamma_L \;=\; s\; B_L\; 2^{-C(L)/2}\;, \quad B_L \;=\; (-1)^{q(L)+1}\;. \tag{3.5}$$

Here, $C(L)$ is the number of corners and $q(L)$ is the number of self-intersections of the loop L. Closely related to this result, free Wilson fermions described by the eqs. (3.1), (3.2) have been shown to be exactly equivalent in 2 dimensions to a two-color loop model with a bending rigidity $\eta = 1/\sqrt{2}$ ([9], also see [10, 11]). For the partition function (3.1) holds in this case

$$Z_\Lambda \;=\; \left(Z_\Lambda\left[M, \frac{1}{\sqrt{2}} \right] \right)^2 \quad , \tag{3.6}$$

where $Z_\Lambda[z,\eta]$ is the partition function for the self-avoiding loop model with monomer weight z and a bending rigidity η as defined in [46]. Within the hopping parameter expansion of the partition function Z_Λ (eq. (3.1)) the equations (2.15), (2.16) can be represented pictorially as shown in fig. 2 (in [13], sect. 4, fig. 2, p. 4860 the relation (2.15) has been dubbed "kink rule", also note [47], sect. 2.2, fig. 3, p. 4558 and eq. (2.16), p. 4557). From this picture one immediately recognizes that $\eta = \cos\Theta = 1/\sqrt{2}$ holds.

For 3 and 4 lattice dimensions, in the appendix of [6] the following result has been obtained. Apply a 4×4 gamma matrix representation ($s = 2$) of the Clifford algebra $C(5,0)$ with

$$\theta_\mu \;=\; \frac{\pi}{4}\;, \quad \mu = 1, 2, 3, 4, \tag{3.7}$$

$$\theta_5 \;=\; 0\;, \tag{3.8}$$

$$w_{k+} \;=\; w'_{k+} \;=\; -i\sigma_k\;, \quad k \;=\; 1, 2, 3, \tag{3.9}$$

$$w_{4+} \;=\; w'_{4+} \;=\; \mathbf{1}_2 \tag{3.10}$$

(this is the so-called chiral representation, see, e.g., [42], appendix 8.1, p. 435; σ_k are the standard Pauli matrices). Then, one finds

$$\operatorname{tr}\Gamma_L \;=\; 2\; B_L\; 2^{-C(L)/2}\;, \tag{3.11}$$

$$B_L \;=\; \frac{1}{2}\;\operatorname{tr}\prod_{l=1}^{C(L)}\;\mathrm{e}^{\left[\frac{\pi}{4}\; w'_{\mu_l k_l} w'^{\dagger}_{\mu_{l+1} l_{l+1}}\right]} \tag{3.12}$$

($\mu_{C(L)+1} = \mu_1$, $l_{C(L)+1} = l_1$; of course, the number of loop sides is equal to the number of corners, we associate the side $\#l$ with the corner $\#l$). This result can easily be related to the discussion performed in sect. 2. Inserting eq. (2.60) into eq. (3.4) and taking into account the eqs. (2.52), (2.46), (2.61) results in

$$\operatorname{tr}\Gamma_L \;=\; \tau^{C(L)/2}\;\operatorname{tr}\prod_{l=1}^{C(L)}\;\check{t}_{(\mu_l k_l,\mu_{l+1} l_{l+1})}\;. \tag{3.13}$$

With the choice (3.7)-(3.10), one immediately recovers from eq. (3.13) the eqs. (3.11), (3.12). Clearly, the eq. (A.3) in the appendix of [6] can be recognized in the eqs. (2.52) and (2.60). Further, in [6] it has been argued that the matrices

$$\mathrm{e}^{\left[\frac{\pi}{4}\; w'_{\mu_l k_l} w'^{\dagger}_{\mu_{l+1} l_{l+1}}\right]} \tag{3.14}$$

belong to a $j = \frac{1}{2}$ representation of the double octahedral group ${}^2O_3$ (for some related review see, e.g., [48]) . The trace in eq. (3.12) is a character of the double octahedral group ${}^2O_3$ and, therefore, one finds that

$$B_L \;=\; \cos\frac{\theta_L}{2}\;. \tag{3.15}$$

θ_L can either be a multiple of π or $\frac{2\pi}{3}$ if $C(L)$ is even, or an odd multiple of $\frac{\pi}{2}$ if $C(L)$ is odd. This can easily be understood by taking recourse to the isomorphy of the octahedral group O_3 to the symmetric group S_4. An even permutation $\in S_4$ corresponds to a rotation of the three-dimensional cube by an angle of π or $\frac{2\pi}{3}$ (classes $3C_2$, $8C_3$, $6C_2'$ of O_3) while an odd permutation corresponds to a rotation by an angle of $\frac{\pi}{2}$ (class $6C_4$ of O_3). Consequently, the matrix (3.14) corresponds to an odd permutation. This fact immediately provides us with the explanation for the mentioned rule (for a more technical argument yielding the same result see the original paper [6]). Furthermore, for loops in 3 lattice dimensions θ_L cannot be a multiple of $\frac{2\pi}{3}$ which is not at the same time a multiple of π.

While eqs. (3.11), (3.12), (3.15) provide us with interesting information, for 3 and 4 lattice dimensions a qualitative insight comparable with eq. (3.5) and generalizing it to these dimensions is lacking so far (some suggestion made for 3 dimensions in [6], p. 1131, can be seen to be incorrect by studying some simple examples). But, as explained in the Introduction it is highly desirable to solve this problem in some satisfactory way or, at least, to make further progress in this direction. One obstacle to gain further insight seems to be that the calculation of B_L according to the eqs. (3.12), (3.15) is related to some non-abelian group (i.e., to the double octahedral group ${}^2O_3$). Consequently, based on the results obtained in sect. 2 in the following we study the question if it is possible to relate the calculation of B_L to some abelian group (or to a group – take this with a grain of salt – exhibiting as little as possible non-abelian structure, or a as simple as possible such structure). For 3 lattice dimensions, we will show in the next subsection that such an abelianization is indeed possible. For 4 dimensions, the situation is considerably more involved and will separately be discussed in subsect. 3.3.

3.2 Dirac traces in 3 lattice dimensions

In 3 lattice dimensions, we can either choose 2×2 ($s = 1$) or 4×4 ($s = 2$) gamma matrices to represent the Clifford algebra $C(3, 0)$. We will start with considering $s = 1$ and derive from this case all necessary information for $s = 2$. To clarify the relation between the Dirac traces taken in the $s = 1$ and $s = 2$ representations of the Clifford algebra $C(3, 0)$ let us start with the following observations (for the moment, we leave s arbitrary). Consider the trace of the matrix (3.4)

$$\mathrm{tr}\,\Gamma_L \;=\; \mathrm{tr} \prod_{l=1}^{|L|} P_{\mu_l k_l} \tag{3.16}$$

and write, say, $P_{\mu_1 k_1}$ in accordance with the formula

$$P_{\mu k} \;=\; \sum_{i=1}^{s} \phi_{(\mu k,i)} \otimes \phi^\dagger_{(\mu k,i)} \tag{3.17}$$

where the set of $\phi_{(\mu k,i)} \in V_{\mu k}$, $(\phi_{(\mu k,i)}, \phi_{(\mu k,j)})_{\mathbf{C}} = \delta_{ij}$, $i, j = 1, \ldots, s$, is an orthonormal system in the subspace of V described by $P_{\mu k}$. Then, we can alternatively write for eq. (3.16)

$$\mathrm{tr}\,\Gamma_L \;=\; \tau^{C(L)/2} \sum_{i=1}^{s} (\phi_{(\mu_1 k_1,i)}, \phi'_{(\mu_1 k_1,i)})_{\mathbf{C}} \;, \tag{3.18}$$

$$\phi'_{(\mu_1 k_1,i)} \;=\; \tau^{-C(L)/2} \prod_{l=1}^{C(L)} P_{\mu_l k_l}\; \phi_{(\mu_1 k_1,i)} \;. \tag{3.19}$$

By virtue of the eqs. (2.15), (2.16), the vectors $\phi'_{(\mu_1 k_1,i)} \in V_{\mu_1 k_1}$, $i = 1, \ldots, s$, also represent an orthonormal system in the subspace $V_{\mu_1 k_1}$, i.e., $(\phi'_{(\mu_1 k_1,i)}, \phi'_{(\mu_1 k_1,j)})_{\mathbf{C}} =$

δ_{ij}. The isocliny of the subspaces described by the projectors $P_{\mu\pm}$ allows us to write eq. (3.18) in the following form.

$$\mathrm{tr}\,\Gamma_L \;=\; s\;\tau^{C(L)/2}\;(\phi_{(\mu_1 k_1,1)},\phi'_{(\mu_1 k_1,1)})_{\mathbf{C}} \tag{3.20}$$

To see that each term in the sum on the r.h.s. of eq. (3.18) contributes the same amount, insert for each projector $P_{\mu_l k_l}$ on the r.h.s. of eq. (3.19) the representation (3.17) and always choose $\phi_{(\mu_{l-1}k_{l-1},i)} = \tau^{-1/2}P_{\mu_{l-1}k_{l-1}}\phi_{(\mu_l k_l,i)}$.

The scalar product $(\phi_{(\mu_1 k_1,1)},\phi'_{(\mu_1 k_1,1)})_{\mathbf{C}}$ in eq. (3.20) is nothing else than the cosine of the complex angle between the (unit) vectors $\phi_{(\mu_1 k_1,1)}$ and $\phi'_{(\mu_1 k_1,1)}$ (for a review on the subject of angles in complex vector spaces and definitions of the angle concepts used here see [29]). First, choose $s=1$. Then, one can write ($\varphi_L \in \mathbf{R}$, $-\pi \le \varphi_L \le \pi$).

$$\phi'_{(\mu_1 k_1,1)} \;=\; \mathrm{e}^{i\varphi_L}\;\phi_{(\mu_1 k_1,1)} \tag{3.21}$$

and, consequently,

$$(\phi_{(\mu_1 k_1,1)},\phi'_{(\mu_1 k_1,1)})_{\mathbf{C}} \;=\; \mathrm{e}^{i\varphi_L}\;. \tag{3.22}$$

φ_L is the pseudo-angle between the vectors $\phi_{(\mu_1 k_1,1)}$ and $\phi'_{(\mu_1 k_1,1)}$. Their Hermitian angle is zero. Thus, eq. (3.20) finally reads for $s=1$

$$\mathrm{tr}\,\Gamma_L \;=\; \tau^{C(L)/2}\;\mathrm{e}^{i\varphi_L}\;. \tag{3.23}$$

Now, choose $s=2$. We apply a real representation of the Clifford algebra $C(3,0)$ by going over from the complex two-dimensional spinor space $V = V_{\mathbf{C}} \simeq \mathbf{C}_2$ for $s=1$ to the corresponding four-dimensional real spinor space $V = V_{\mathbf{R}} \simeq \mathbf{R}_4$ for $s=2$ which is isometric to the former. In accordance with the almost complex structure defined in $V_{\mathbf{R}}$ by means of eq. (2.17), a vector $\Phi \in V_{\mathbf{R}}$ is related to a vector $\phi = (\phi_1,\phi_2)^T \in V_{\mathbf{C}}$ by the formula

$$\Phi^T \;=\; (\mathrm{Re}\;\phi_1, \mathrm{Im}\;\phi_1, \mathrm{Re}\;\phi_2, \mathrm{Im}\;\phi_2)\;. \tag{3.24}$$

As the eigenspaces of the gamma matrices in the chosen real representation of the Clifford algebra $C(3,0)$ are holomorphic 2-planes (cf. eq. (3.21)), their Kähler angle is zero and, consequently, the (Euclidean) angle between the vectors $\Phi_{(\mu_1 k_1,1)}$ and $\Phi'_{(\mu_1 k_1,1)}$ is equal to their pseudo-angle (cf. sect. 4 of [29]). Thus, for $s=2$ eq. (3.20) reads

$$\mathrm{tr}\,\Gamma_L \;=\; 2\;\tau^{C(L)/2}\;\cos\varphi_L\;. \tag{3.25}$$

From eq. (3.15) we recognize that $\varphi_L = \theta_L/2$. The eqs. (3.23) and (3.25) provide us with the precise relation between the Dirac traces taken in the $s=1$ and the $s=2$

representations of the Clifford algebra $C(3,0)$. Therefore, in the remainder of this subsection for simplicity we confine our consideration to the case $s = 1$.

We first study eq. (2.50). In view of eq. (2.43), we set $k, l = +(1)$. For $s = 1$, we can write $w_{\mu+}$, $\mu = 1, 2, 3$, as follows (cf. eq. (2.44); $\beta_\mu \in \mathbf{R}$).

$$w_{\mu+} \;=\; \sin 2\theta_\mu \; \mathrm{e}^{i\beta_\mu} \tag{3.26}$$

This leads to these three equations ($\Delta\beta_{\mu\nu} = \beta_\mu - \beta_\nu$; we assume here $\theta_\mu \neq 0, \frac{\pi}{2}$, these cases have to be studied separately).

$$\cos\Delta\beta_{12} \;=\; -\; \cot 2\theta_1 \; \cot 2\theta_2 \tag{3.27}$$

$$\cos\Delta\beta_{13} \;=\; -\; \cot 2\theta_1 \; \cot 2\theta_3 \tag{3.28}$$

$$\cos\Delta\beta_{23} \;=\; -\; \cot 2\theta_2 \; \cot 2\theta_3 \tag{3.29}$$

The system of eqs. (3.27)-(3.29) can be transformed to read

$$\cot^2 2\theta_1 \;=\; -\; \frac{\cos\Delta\beta_{12}\; \cos\Delta\beta_{13}}{\cos\Delta\beta_{23}}\;, \tag{3.30}$$

$$\cot^2 2\theta_2 \;=\; -\; \frac{\cos\Delta\beta_{12}\; \cos\Delta\beta_{23}}{\cos\Delta\beta_{13}}\;, \tag{3.31}$$

$$\cot^2 2\theta_3 \;=\; -\; \frac{\cos\Delta\beta_{23}\; \cos\Delta\beta_{13}}{\cos\Delta\beta_{12}}\;. \tag{3.32}$$

One can convince oneself by explicit calculation that for any (admissible) choice of β_μ, $\mu = 1, 2, 3$, the isocliny angles θ_μ given by the eqs. (3.30)-(3.32) respect the constraint eq. (2.18) ($p = 3$). Conversely, any (admissible) choice of the isocliny angles θ_μ determines β_μ (up to a μ independent constant and other obvious symmetries of the eqs. (3.27)-(3.29)). Inserting eq. (3.26) into eq. (2.63) and taking into account eqs. (3.27)-(3.29) we find

$$I_{(\mu k, \nu l)} \;=\; i\;. \tag{3.33}$$

To arrive at this result,

$$f_{\mu\nu} \;=\; \mathrm{sgn}\left[\sin\left(\beta_\mu - \beta_\nu\right)\right] \tag{3.34}$$

has been chosen. In the following, we will consider two different special solutions of the eqs. (3.27)-(3.29). The first one we will refer to as the symmetric case while we will call the second one the Pauli case.

The symmetric case is approached as follows. For our purpose of calculating Dirac traces within lattice quantum field theory, it seems to be most natural and appropriate to choose β_μ, θ_μ in a most symmetric way. Therefore, taking into account eq. (2.18) we set

$$\theta_1 \;=\; \theta_2 \;=\; \theta_3 \;=\; \theta_{\rm sym}^{\{\pm\}} \;, \quad \cos 2\theta_{\rm sym}^{\{\pm\}} \;=\; \pm\frac{1}{\sqrt{3}} \;. \tag{3.35}$$

This choice entails (in both cases)

$$\beta_\mu \;=\; \beta(\mu) \;=\; \beta_0 \;+\; \frac{2\pi}{3}\;\mu \tag{3.36}$$

($\beta_0 \in \mathbf{R}$ is some arbitrary constant). Inserting this choice into eq. (2.45) (select $\theta_{\rm sym}^{\{+\}}$), we immediately see that these gamma matrices agree (up to complex conjugation) with the transformed Pauli matrices found in [4], sect. I, eq. (11), p. 3618. Proceeding further, from eqs. (2.63)-(2.65), (3.35) we find[2]

$$\alpha^{\{\pm\}}_{(\mu+,\nu+)} \;=\; f_{\mu\nu}\;\left\{\begin{array}{c} 1 \\ 7 \end{array}\right\}\frac{\pi}{12}\;, \tag{3.37}$$

$$\alpha^{\{\pm\}}_{(\mu+,\nu-)} \;=\; \alpha^{\{\pm\}}_{(\mu-,\nu+)} \;=\; -\;f_{\mu\nu}\;\frac{\pi}{6}\;, \tag{3.38}$$

$$\alpha^{\{\pm\}}_{(\mu-,\nu-)} \;=\; f_{\mu\nu}\;\left\{\begin{array}{c} 7 \\ 1 \end{array}\right\}\frac{\pi}{12}\;. \tag{3.39}$$

Consequently, we can express φ_L as follows.

$$\varphi_L \;=\; \frac{\theta_L}{2} \;\equiv\; \left(\sum_{l=1}^{C(L)}\;\alpha^{\{\pm\}}_{(\mu_l k_l,\mu_{l+1}k_{l+1})}\right)\mathrm{mod}(2\pi) \tag{3.40}$$

In other words, each loop corner occurring in the hopping parameter expansion of Wilson fermions in 3 lattice dimensions can be attached an angle $\alpha^{\{\pm\}}_{(\mu k,\nu l)}$ whose linear sum determines the value of $\mathrm{tr}\,\Gamma_L$. The abelian character of this sum should be ideally suited for any future study of the generalization of eq. (3.5) to 3 lattice dimensions (the same comment applies to the Pauli case below).

We define the Pauli case by choosing

$$\gamma_1 \;=\; \sigma_1 \;=\; \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right) \quad (3.41) \qquad\qquad \gamma_2 \;=\; \sigma_2 \;=\; \left(\begin{array}{cc} 0 & -i \\ i & 0 \end{array}\right) \quad (3.42)$$

[2] Incidentally, the appearance of $\frac{\pi}{12} = \frac{2\pi}{24}$ as value of the angles $\alpha^{\{\pm\}}_{(\mu k,\nu l)}$ is eye-catching to some extent as this angle does not occur very often in mathematics. One is immediately lead to think of the Dedekind η function where it also occurs. If this signals more than an accidental coincidence has to remain open for now.

$$\gamma_3 \;=\; \sigma_3 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.43}$$

where σ_k, $k = 1, 2, 3$, are the standard Pauli matrices. This choice entails

$$\theta_1 \;=\; \frac{\pi}{4}\,, \quad w_{1+} \;=\; 1\,, \quad \beta_1 \;=\; 0\,, \tag{3.44}$$

$$\theta_2 \;=\; \frac{\pi}{4}\,, \quad w_{2+} \;=\; -i\,, \quad \beta_2 \;=\; -\frac{\pi}{2}\,, \tag{3.45}$$

$$\theta_3 \;=\; 0\,, \quad w_{3+} \;=\; 0\,. \tag{3.46}$$

Of course, β_3 remains undetermined and U_{3-} is not immediately given by eq. (2.52), only after some closer inspection. We find

$$U_{3-} \;=\; \begin{pmatrix} 0 & -\,\mathrm{e}^{i\beta_3} \\ \mathrm{e}^{-i\beta_3} & 0 \end{pmatrix}\,. \tag{3.47}$$

The matrices $\check{t}_{(\mu k,\nu l)}$ can again be found without any problem from eqs. (2.62)-(2.65), except for $\mu = 3$, $k = -(1)$ (or $\nu = 3$, $l = -(1)$). To proceed in the latter case on the basis of the eqs. (2.62)-(2.65) involves a subtle limiting procedure for the isocliny angles $\theta_{\mu k}$. The limit for θ_1, θ_2, θ_3 towards $\frac{\pi}{4}$ and 0 is ruled by the constraint eq. (2.18) ($p = 3$) and the value of β_3 has to be adjusted in accordance with the chosen approach towards this limit on the basis of the eqs. (3.28), (3.29). For $\mu = 3$, $k = -(1)$ (or $\nu = 3$, $l = -(1)$), a direct calculation is simpler and yields ($\nu = 1, 2$)

$$\check{t}_{(3-,\nu l)} \;=\; \check{t}^{\dagger}_{(\nu l,3-)} \;=\; \sqrt{2}\,\left[U_{3-}U^{\dagger}_{\nu l}\right]_{11} \;=\; -\,l\;\mathrm{e}^{i(\beta_3\,-\,\beta_\nu)}\,. \tag{3.48}$$

Finally, we find (in the present case we have chosen $f_{12} = f_{13} = f_{23} = 1$)

$$\alpha_{(1k,2l)} \;=\; -\,\alpha_{(2l,1k)} \;=\; k\;l\;f_{12}\,\frac{\pi}{4}\,, \tag{3.49}$$

$$\alpha_{(\nu l,3+)} \;=\; \alpha_{(3+,\nu l)} \;=\; 0\,, \tag{3.50}$$

$$\alpha_{(\nu+,3-)} \;=\; -\,\alpha_{(3-,\nu+)} \;=\; f_{\nu 3}\,(\beta_\nu - \beta_3 + \pi)\,, \tag{3.51}$$

$$\alpha_{(\nu-,3-)} \;=\; -\,\alpha_{(3-,\nu-)} \;=\; f_{\nu 3}\,(\beta_\nu - \beta_3)\,. \tag{3.52}$$

Again, φ_L can be expressed the same way as in eq. (3.40).

$$\varphi_L \;=\; \frac{\theta_L}{2} \;\equiv\; \left(\sum_{l=1}^{C(L)} \alpha_{(\mu_l k_l,\mu_{l+1}k_{l+1})}\right)\mathrm{mod}(2\pi) \tag{3.53}$$

For simplicity, we can set $\beta_3 = 0$. Then, eqs. (3.49)-(3.53) immediately tell us that in 3 lattice dimension θ_L must be a multiple of $\frac{\pi}{2}$ as argued in [6] by other means.

Also, a another rule given in [6] can independently be rederived (also cf. the comments made in the present paper below of eq. (3.15)). Replace in Γ_L (eq. (3.4)) all projectors $P_{2\pm}$ by projectors $P_{1\pm}$ (it does not matter for the present argument if the procedure makes any geometrical sense). Then, for the modified trace $\mathrm{tr}\,\Gamma_{L'}$ the corner number $C(L')$ is always even and all corner angles are given by eqs. (3.50)-(3.52). Consequently, $\theta_{L'}$ is an even multiple of $\frac{\pi}{2}$. Therefore, the number of corners with $\mu_l = 1$, $\mu_{l+1} = 2$, or $\mu_l = 2$, $\mu_{l+1} = 1$, determines if $C(L)$ is even or odd. From this fact and eq. (3.49) we can immediately conclude that θ_L can only be an odd multiple of $\frac{\pi}{2}$ if the corner number $C(L)$ is also odd.

Finally, let us shortly comment on the Dirac traces taken in the $s = 2$ representation of the Clifford algebra $C(3,0)$. According to eq. (2.17), eq. (3.33) then reads

$$I_{(\mu k,\nu l)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} . \tag{3.54}$$

Consequently, within the present approach the matrices (2.62) related to the loop corners are $SO(2)$ matrices (σ_2 denotes the second Pauli matrix)

$$\check{t}_{(\mu k,\nu l)} = \begin{pmatrix} \cos\alpha_{(\mu k,\nu l)} & -\sin\alpha_{(\mu k,\nu l)} \\ \sin\alpha_{(\mu k,\nu l)} & \cos\alpha_{(\mu k,\nu l)} \end{pmatrix} = \mathrm{e}^{-i\alpha_{(\mu k,\nu l)}\sigma_2} . \tag{3.55}$$

For $s = 2$, the gamma matrices given by eq. (2.45) agree in the symmetric case (up to some elementary transformation, i.e., an inversion) with those given in [4], sect. I, eq. (7), p. 3617.

3.3 Dirac traces in 4 lattice dimensions

Unfortunately, the strategy applied in the previous subsection for 3 lattice dimensions cannot be extended to 4 dimensions. The reason for this consists in the fact that there is no $s = 1$ matrix representation of the Clifford algebra $C(4,0)$. On a technical level, in our approach this fact raises its head if one would attempt to solve the analogue of eqs. (3.27)-(3.29) for 4 dimensions. This set of equations would then consist of 6 equations which are no longer simultaneously solvable. Consequently, in 4 lattice dimensions we have to work right with a $s = 2$ representation of the gamma matrices. Now, what is the best strategy in 4 lattice dimensions? We do not have any final answer on this question but it seems not unreasonable to assume for the moment that the approach should be based on the results obtained in the previous subsection for 3 dimensions.

First, we have to study eq. (2.50) again. As in the chosen approach we have in accordance with eq. (3.26)

$$w_{\mu +} \;=\; \sin 2\theta_\mu \begin{pmatrix} \cos\beta_\mu & -\sin\beta_\mu \\ \sin\beta_\mu & \cos\beta_\mu \end{pmatrix} \;, \qquad \mu = 1,2,3, \tag{3.56}$$

(in particular, this equation applies in both the symmetric and the Pauli cases but not only then) we immediately find from eq. (2.50) (clearly, in view of eq. (2.18) $\theta_4 = \frac{\pi}{4}$)

$$w_{4+} \;=\; i \begin{pmatrix} \cos\beta_4 & \sin\beta_4 \\ \sin\beta_4 & -\cos\beta_4 \end{pmatrix} \;. \tag{3.57}$$

Consequently, the matrix $-iw_{4+}$ belongs to the group $O(2)$. $\beta_4 \in \mathbf{R}$ can freely be chosen. From eqs. (2.62)-(2.65) we find (σ_2, σ_3 are the standard Pauli matrices)

$$\begin{aligned}
\check{t}_{(\mu k,4l)} &= \check{t}^\dagger_{(4l,\mu k)} \\
&= \cos\theta_{\mu k} - i\sin\theta_{\mu k} \begin{pmatrix} \cos(\beta_\mu+\beta_4) & \sin(\beta_\mu+\beta_4) \\ \sin(\beta_\mu+\beta_4) & -\cos(\beta_\mu+\beta_4) \end{pmatrix} \qquad (3.58) \\
&= \begin{pmatrix} \cos\frac{\beta_\mu+\beta_4}{2} & -\sin\frac{\beta_\mu+\beta_4}{2} \\ \sin\frac{\beta_\mu+\beta_4}{2} & \cos\frac{\beta_\mu+\beta_4}{2} \end{pmatrix} \begin{pmatrix} \mathrm{e}^{-i\theta_{\mu k}} & 0 \\ 0 & \mathrm{e}^{i\theta_{\mu k}} \end{pmatrix} \\
&\quad \times \begin{pmatrix} \cos\frac{\beta_\mu+\beta_4}{2} & \sin\frac{\beta_\mu+\beta_4}{2} \\ -\sin\frac{\beta_\mu+\beta_4}{2} & \cos\frac{\beta_\mu+\beta_4}{2} \end{pmatrix} \\
&= \exp\left[-i\,\frac{\beta_\mu+\beta_4}{2}\,\sigma_2\right]\,\exp\left[-i\theta_{\mu k}\sigma_3\right]\,\exp\left[i\,\frac{\beta_\mu+\beta_4}{2}\,\sigma_2\right] \;. \qquad (3.59)
\end{aligned}$$

The last line can be understood as a representation of the $SU(2)$ matrix $\check{t}_{(\mu k,4l)}$ in terms of Euler angles (cf., e.g., [49], chap. 2, sect. 7, eq. (2.40), p. 24). We can then write

$$\begin{aligned}
\check{t}_{(\mu k,4l)}\check{t}_{(4l,\nu m)} &= \exp\left[-i\,\frac{\beta_\mu+\beta_4}{2}\,\sigma_2\right]\,\exp\left[-i\theta_{\mu k}\sigma_3\right]\,\exp\left[i\,\frac{\beta_\mu-\beta_\nu}{2}\,\sigma_2\right] \\
&\quad \times\,\exp\left[i\theta_{\nu m}\sigma_3\right]\,\exp\left[i\,\frac{\beta_\nu+\beta_4}{2}\,\sigma_2\right] \;. \qquad (3.60)
\end{aligned}$$

From the above equations one recognizes that in 4 lattice dimensions the non-abelian character of the product appearing on the r.h.s. of eq. (3.13) is closely related to the values of the isocliny angles $\theta_{\mu k}$. Introducing the notation (we assume without restricting generality $\theta_{\mu +} \leq \frac{\pi}{4}$)

$$\theta_{\mu k} \;=\; \frac{\pi}{4} \;-\; k\;\Delta\theta_\mu \tag{3.61}$$

we can express the product of the three middle matrices on the r.h.s. of eq. (3.60) which encodes the non-abelian structure as follows (here, $\omega_{\mu\nu} = (\beta_\mu - \beta_\nu)/2$).

$$\begin{aligned}
\exp\left[-i\theta_{\mu k}\sigma_3\right]\ \exp\left[i\ \omega_{\mu\nu}\ \sigma_2\right]\ \exp\left[i\theta_{\nu m}\sigma_3\right] & \\
= \ & \cos\left(k\Delta\theta_\mu - m\Delta\theta_\nu\right)\ \cos\omega_{\mu\nu} \\
& - i\sigma_1\ \sin\omega_{\mu\nu}\ \cos\left(k\Delta\theta_\mu + m\Delta\theta_\nu\right) \\
& + i\sigma_2\ \sin\omega_{\mu\nu}\ \sin\left(k\Delta\theta_\mu + m\Delta\theta_\nu\right) \\
& + i\sigma_3\ \cos\omega_{\mu\nu}\ \sin\left(k\Delta\theta_\mu - m\Delta\theta_\nu\right)
\end{aligned} \tag{3.62}$$

From this expression, one easily recognizes that it would be advantageous if $\Delta\theta_\mu = 0$ (i.e., $\theta_{\mu k} = \frac{\pi}{4}$) applied for as many as possible values of μ in order to facilitate and to simplify the further study of Dirac traces in 4 lattice dimensions. It is clear, that in view of eq. (2.18) $\Delta\theta_1 = \Delta\theta_2 = \Delta\theta_3 = 0$ is not an admissible choice. There are two alternatives to this best, but inadmissible choice. Either one applies the condition $\Delta\theta_1 = \Delta\theta_2 = \Delta\theta_3$ ($\neq 0$) which leads to the symmetric case or, one chooses $\Delta\theta_1 = \Delta\theta_2 = 0$ which leads to the Pauli case (also, considering $\omega_{\mu\nu}$ does not lead to any better choices as the values of $\omega_{\mu\nu}$ are closely related to the isocliny angles θ_μ according to eqs. (3.27)-(3.27)). Studying in both cases the explicit matrix expressions (3.62), unfortunately neither the symmetric nor the Pauli case seem to exhibit any particular advantage for their application in the further study of Dirac traces in 4 lattice dimensions. Still, alternatively one might try to further uphold the idea that possibly best further progress could be made if all eight isocliny angles $\theta_{\mu\pm}$ one has to deal with in 4 dimensions would have the same value. This can only be achieved by extending the consideration in the Pauli case to the Clifford algebra $C(5,0)$ (which is the maximal Clifford algebra with a $s=2$ representation). Then, eight isocliny angles $\theta_{\mu\pm}$ can assume the value $\frac{\pi}{4}$. From eq. (3.57) we immediately find (again, eq. (2.18) tells us that $\theta_5 = \frac{\pi}{4}$)

$$w_{5+} \ = \ i \begin{pmatrix} \cos\beta_5 & \sin\beta_5 \\ \sin\beta_5 & -\cos\beta_5 \end{pmatrix} \tag{3.63}$$

and eq. (2.50) (which agrees in the Pauli case with eq. (2.58)) yields

$$\beta_4 - \beta_5 \ \equiv \ \frac{\pi}{2}\,\mathrm{mod}(\pi)\ . \tag{3.64}$$

To be specific, we choose

$$\beta_4 \ = \ 0\ , \tag{3.65}$$

$$\beta_5 \ = \ \frac{\pi}{2}\ . \tag{3.66}$$

Our choices result in the following $s=2$ representation of the Clifford algebra $C(4,0)$ ($\theta_{1\pm} = \theta_{2\pm} = \theta_{4\pm} = \theta_{5\pm} = \frac{\pi}{4}$). We only display the matrices $w_{\mu+}$, for the corresponding gamma matrices see eq. (2.45) (σ_k are the standard Pauli matrices).

$$w_{1+} = \mathbf{1}_2 \tag{3.67}$$

$$w_{2+} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i\sigma_2 \tag{3.68}$$

$$w_{4+} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\sigma_3 \tag{3.69}$$

$$w_{5+} = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = i\sigma_1 \tag{3.70}$$

Obviously, this consideration has brought us back to the chiral representation of the gamma matrices which had already been used in [6] (cf. eqs. (3.7)-(3.10)). At the level of the present study, it seems difficult to make any final judgement which representation of Dirac traces for Wilson fermions in 4 lattice dimensions will facilitate their further study in the most effective way. Consequently, the current subsection could be best understood as an explorative one.

4 Final comments

The present study has revealed that there is a considerable difference between Dirac traces for Wilson fermions in 3 and 4 lattice dimensions. While in 3 dimensions each loop corner occurring in the hopping parameter expansion for Wilson fermions an angle can be associated with whose linear sum (over all loop corners) determines the value of the corresponding trace such a simple procedure cannot be introduced in 4 lattice dimensions. This follows in a quite straightforward way from the study of Clifford algebra representations performed in sect. 2. On the basis of the parametrization of gamma matrices in terms of isocliny angles, we have discussed some of the possible representations of Dirac traces in 4 dimensions. The formalism developed in the present paper allows in principle to systematically map out the space of such representations and, therefore, should be useful in any future study of the subject. It remains to be hoped that the progress achieved in the present paper will not only allow to gain further future understanding of Dirac traces for Wilson fermions in 3 lattice dimensions but also to successfully attack the problem in 4 dimensions.

At the end of this study, we would like to mention some further related articles. Wilson fermions can be understood as a statistical system with matrix-valued ($SU(2)$) vertex weights (and a bending rigidity $\eta = 1/\sqrt{2}$; cf. eqs. (3.3), (3.13)). Models of the same type have been considered in the past, e.g., in the approximate study of the three-dimensional Ising model [50]-[52] (in these papers the authors apply $SU(2)$ vertex weights and a bending rigidity $\eta = 1$). A variable bending rigidity has been studied in a four-dimensional lattice fermion model with (Wilson fermion) $SU(2)$ matrix vertex weights in [53, 54] ('link fermions'). Also, recently a three-dimensional loop model with $SU(2)$ vertex weights and a variable bending rigidity η has been considered [55] (however, the authors of this work pay special attention to the bending rigidity $\eta = 1/\sqrt{2}$ which is the case most closely related to Wilson fermions; in this context it should be pointed out that only further research can show how eq. (3.6) is properly generalized to 3 lattice dimensions). Finally, we want to mention that other three-dimensional loop models with loop shape dependent weights have also been considered [56]-[58].

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